Name: RED ID:

Spring 2024 Math 245 Exam 2

Please read the following directions:

Please write legibly, with plenty of white space. Please **print** your name and REDID in the designated spaces above. Please fit your answers into the designated areas; material outside the designated areas (such as on this cover page) will not be graded. To get credit, you must also show adequate work to justify your answers. If unsure, show the work. All problems are worth 6-12 points, for a minimum score of 60/120 and a maximum score of 120/120. The use of notes, books, calculators, or other materials on this exam is strictly prohibited, except you may bring one 3"x5" card (both sides) with your handwritten notes. This exam will begin at 10:00 and will end at 10:50; pace yourself accordingly. Good luck!

Special exam instructions for SSW-1500:
1. Please don't leave empty seats in the middle of the classroom – they will only get filled after the exam starts, which is better for nobody. As usual, please leave the seats on the far right for latecomers.
2. Please stow all bags/backpacks/boards at the front of the room. All contraband, except phones, must be stowed in your bag. All smartwatches and phones must be silent, non-vibrating, and either in your pocket or stowed in your bag.
3. Please remain quiet to ensure a good test environment for others.
4. Please keep your exam on your desk; do not lift it up for a better look.
5. If you have a question or need to use the restroom, please come to the front. Bring your exam. I cannot come to you unless you are sitting by an aisle, sorry.
6. If you are done and want to submit your exam and leave, please wait until one of the three designated exit times, listed below. Please do **NOT** leave at any other time. If you are sure you are done, just sit and wait until the next exit time, with this cover sheet visible.

Designated exam exit times:
- 10:20 "I need to work harder"
- 10:40 "I can't wait to get out of here"
- 10:50 "I need every second I can get"

REMINDER: Use complete sentences.

Problem 1. Carefully define the following terms:

a. well-ordered (for sets)

b. recurrence

Problem 2. Carefully state the following theorems:

a. Nonconstructive Existence Theorem

b. (Vanilla) Induction Theorem

Problem 3. Let $a_n = 7n^2 + 3$. Prove that $a_n = \Theta(n^2)$.

Problem 4. Prove or disprove: For all $x \in \mathbb{R}$, $\lceil x \rceil \leq \lfloor x \rfloor + 1$.

Problem 5. Prove or disprove: For all $n \in \mathbb{Z}$, the number $\frac{n(n-1)(n-2)(n-4)}{5}$ is an integer.

Problem 6. Suppose that an algorithm has runtime specified by recurrence relation $T_n = 3T_{n/2} + n^2$. Determine what, if anything, the Master Theorem tells us.

Problem 7. Let $n \in \mathbb{Z}$. Prove that the following are equivalent: (a) n is even; (b) $7n$ is even; (c) $n + 1$ is odd.

For problems 8-10, we fix unknown positive real numbers r, s, t, u, and consider the recurrence given by $x_1 = r, x_2 = s$, and $x_n = tx_{n-1} + ux_{n-2}$ (for $n \geq 3$).

Problem 8. Prove that if $\{a_n\}$ satisfies the recurrence then $a_n \leq (2M)^n$ (for all $n \in \mathbb{N}$), where $M = \max(r, s, t, u, 1)$.

Problem 9. Prove that there exists at least one sequence $\{a_n\}$ satisfying this recurrence (with indices in $\mathbb{N}$, i.e. $n \in \mathbb{N}$). NOTE: Just prove it exists, do not try to find a closed form.

Problem 10. Prove that there exists at most one sequence $\{a_n\}$ satisfying this recurrence (with indices in $\mathbb{N}$, i.e. $n \in \mathbb{N}$). HINT: minimum element induction on a set of indices.